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Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense

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ABSTRACT

New inequalities of Ostrowski type for functions whose derivatives in absolute value are s-convex in the second sense are obtained.

1. Introduction

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $\left|f'(x)\right| \le M$, then the following inequality:

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{M}{b-a}\cdot\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] \tag{1.1}$$

holds. This result is known in the literature as the *Ostrowski inequality*. For recent results and generalizations concerning Ostrowski's inequality see [1–7] and the references therein.

In [8], Hudzik and Maligranda considered among others the class of functions which are *s-convex in the second sense*. This class is defined in the following way: a function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions is usually denoted by K_s^2. It can be easily seen that for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$.

In [9], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense:

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Theorem 1. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L^1[a,b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2). The above inequalities are sharp.

For recent results and generalizations concerning s-convex functions see [6,7,9,10].

The aim of this work is to establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense.

2. Ostrowski type inequalities

In order to prove our main theorems, we need the following lemma that has been obtained in [3]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u = \frac{(x-a)^2}{b-a}\int_0^1 t f'(tx+(1-t)a)\,\mathrm{d}t - \frac{(b-x)^2}{b-a}\int_0^1 t f'(tx+(1-t)b)\,\mathrm{d}t$$

for each $x \in [a,b]$.

A simple proof of the equality can be given by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

The following result may be stated:

Theorem 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \le M$, $x \in [a,b]$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{M}{b-a}\cdot\left[\frac{(x-a)^2+(b-x)^2}{s+1}\right], \tag{2.1}$$

for each $x \in [a,b]$.

Proof. By Lemma 1 and since $|f'|$ is s-convex, then we have

$$\begin{aligned}
\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|\mathrm{d}t + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|\mathrm{d}t \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t\left[t^s\left|f'(x)\right| + (1-t)^s\left|f'(a)\right|\right]\mathrm{d}t + \frac{(b-x)^2}{b-a}\int_0^1 t\left[t^s\left|f'(x)\right| + (1-t)^s\left|f'(b)\right|\right]\mathrm{d}t \\
&\le \frac{M(x-a)^2}{b-a}\left(\frac{1}{s+2} + \frac{1}{(s+1)(s+2)}\right) + \frac{M(b-x)^2}{b-a}\left(\frac{1}{s+2} + \frac{1}{(s+1)(s+2)}\right) \\
&= \frac{M}{b-a}\cdot\left[\frac{(x-a)^2+(b-x)^2}{s+1}\right],
\end{aligned}$$

where we have used the fact that

$$\int_0^1 t^{s+1}\,\mathrm{d}t = \frac{1}{s+2} \quad \text{and} \quad \int_0^1 t(1-t)^s\,\mathrm{d}t = \frac{1}{(s+1)(s+2)}.$$

This completes the proof. □

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \le M$, $x \in [a,b]$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \tag{2.2}$$

for each $x \in [a,b]$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| &\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|\mathrm{d}t\\ &\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is s-convex in the second sense and $|f'(x)| \le M$, then we have

$$\begin{aligned}\int_0^1\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t &\le \int_0^1\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]\mathrm{d}t\\ &= \frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\le\frac{2M^q}{s+1}\end{aligned}$$

and

$$\begin{aligned}\int_0^1\left|f'(tx+(1-t)b)\right|^q\mathrm{d}t &\le \int_0^1\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]\mathrm{d}t\\ &= \frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\le\frac{2M^q}{s+1}.\end{aligned}$$

Therefore, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right|\le\frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],$$

where $1/p + 1/q = 1$, which is required. □

A different approach leads to the following result:

Theorem 4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, and $|f'(x)| \le M$, $x \in [a,b]$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right|\le M\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\tag{2.3}$$

for each $x \in [a,b]$.

Proof. Suppose that $q \ge 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| &\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|\mathrm{d}t\\ &\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is s-convex, we have

$$\begin{aligned}\int_0^1 t\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t &\le \int_0^1\left[t^{s+1}\left|f'(x)\right|^q+t(1-t)^s\left|f'(a)\right|^q\right]\mathrm{d}t\\ &= \frac{(s+1)\left|f'(x)\right|^q+\left|f'(a)\right|^q}{(s+1)(s+2)}\le\frac{M^q}{s+1}\end{aligned}$$

and

$$\int_0^1 t \left|f'(tx+(1-t)b)\right|^q \mathrm{d}t \le \int_0^1 \left[t^{s+1}\left|f'(x)\right|^q + t(1-t)^s\left|f'(b)\right|^q\right]\mathrm{d}t$$
$$= \frac{(s+1)\left|f'(x)\right|^q + \left|f'(b)\right|^q}{(s+1)(s+2)} \le \frac{M^q}{s+1}.$$

Therefore, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]$$

which is required. □

Remark 1. Since $(1+p)^{\frac{1}{p}} < 2$ for any $p > 1$, then we observe that the inequality (2.3) is better than the inequality (2.2) meaning that the approach via the power mean inequality is a better approach than that through Hölder's inequality.

Remark 2. 1. In the previous inequalities, one can obtain several midpoint type inequalities by setting $x = \frac{a+b}{2}$. However, the details are left to the interested reader.

2. All of the above inequalities obviously hold for convex functions. Simply choose $s = 1$ in each of those results to get the desired results.

The following result holds for s-concavity:

Theorem 5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is s-concave on $[a,b]p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right| + (b-x)^2\left|f'\left(\frac{b+x}{2}\right)\right|\right], \tag{2.4}$$

for each $x \in [a,b]$.

Proof. Suppose that $q > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|\mathrm{d}t + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|\mathrm{d}t$$
$$\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t\right)^{1/q}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q\mathrm{d}t\right)^{1/q}. \tag{2.5}$$

But since $\left|f'\right|^q$ is s-concave, using the inequality (1.2), we have

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q\mathrm{d}t \le 2^{s-1}\left|f'\left(\frac{x+a}{2}\right)\right|^q, \tag{2.6}$$

and

$$\int_0^1\left|f'(tx+(1-t)b)\right|^q\mathrm{d}t \le 2^{s-1}\left|f'\left(\frac{b+x}{2}\right)\right|^q. \tag{2.7}$$

By combining the above numbered inequalities, we get

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{2^{(s-1)/q}}{(1+p)^{1/p}(b-a)}\left[(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right| + (b-x)^2\left|f'\left(\frac{b+x}{2}\right)\right|\right]$$

This completes the proof. □

A midpoint type inequality for functions whose derivatives in absolute value are s-concave in the second sense may be obtained from the previous result as follows:

Corollary 1. *If for (2.4) we choose $x = \frac{a+b}{2}$, then we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{2^{(s-1)/q}(b-a)}{4(1+p)^{1/p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \tag{2.8}$$

For instance, if $s = 1$, *then we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u\right| \le \frac{(b-a)}{4(1+p)^{1/p}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \tag{2.9}$$

where $\left|f'\right|^q$ *is s-concave on* $[a, b]$, $p > 1$.

3. Applications to special means

We consider the means for arbitrary positive numbers $\alpha, \beta (\alpha \neq \beta)$ as follows:

1. The arithmetic mean:

$$A(\alpha, \beta) = \frac{\alpha+\beta}{2},$$

2. The generalized log-mean:

$$L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

3. The identric mean:

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta, \end{cases} \qquad \alpha, \beta > 0$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

In [9], the following example is given: Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define a function $f : [0, \infty) \to \mathbb{R}$, as

$$f(t) = \begin{cases} a, & t = 0 \\ bt^s + c, & t > 0. \end{cases}$$

If $b \ge 0$ and $0 \le c \le a$, then $f \in K_s^2$. Hence, for $a = c = 0, b = 1$, we have $f : [0, 1] \to [0, 1], f(t) = t^s, f \in K_s^2$.

Proposition 1. *Let* $0 < a < b$ *and* $0 < s < 1$. *Then we have*

$$\left|A^s(a, b) - L_s^s(a, b)\right| \le \frac{M(b-a)}{4}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}, \quad q \ge 1.$$

Proof. The inequality follows from (2.3) with $x = \frac{a+b}{2}$ applied to the s-convex function in the second sense $f : [0, 1] \to [0, 1], f(x) = x^s$. The details are omitted. □

Proposition 2. *Let* $0 < a < b$ *and* $p > 1$. *Then we have*

$$|\ln A(a, b) - \ln I(a, b)| \le \frac{(b-a)}{(1+p)^{1/p}}\left[\frac{1}{3a+b} + \frac{1}{a+3b}\right], \quad p > 1.$$

Proof. The inequality follows from (2.9) applied to the concave function in the second sense $f : [a, b] \to \mathbb{R}, f(x) = \ln x$. The details are omitted. □

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